The Euler-Lagrange and Hamilton-Jacobi actions and the principle of least action

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Abstract

We recall the main properties of the classical action of Euler-Lagrange $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$, which links the initial position $\mathbf{x}_0$ and its position $\mathbf{x}$ at time t, and of the Hamilton-Jacobi action, which connects a family of particles of initial action $S_0(\mathbf{x})$ to their various positions $\mathbf{x}$ at time t.

Mathematically, the Euler-Lagrange action can be considered as the elementary solution of the Hamilton-Jacobi equation in a new branch of nonlinear mathematics, the Minplus analysis. Physically, we show that, contrary to the Euler-Lagrange action, the Hamilton-Jacobi action satisfies the principle of least action. It is a clear answer on the interpretation of this principle. Finally, we use the relation-ship between the Hamilton-Jacobi and Euler-Lagrange actions to study the convergence of quantum mechanics, when the Planck constant tends to 0, for a particular class of quantum systems, the statistical semiclassical case.

I. INTRODUCTION

In 1744, Pierre-Louis Moreau de Maupertuis (1698-1759) introduced the action and the principle of least action into classical mechanics:[1] "*Nature, in the production of its effects, does so always by the simplest means [...] the path it takes is the one by which the quantity of action is the least,*" and in 1746, he states:[2] "*This is the principle of least action, a principle so wise and so worthy of the supreme Being, and intrinsic to all natural phenomena [...] When a change occurs in Nature, the quantity of action necessary for change is the smallest possible. The quantity of action is the product of the mass of the body times its velocity and the distance it moves.*" Maupertuis understood that, under certain conditions, Newton's equations are equivalent to the fact that a quantity, which he calls the action, is minimal. Euler,[3] Lagrange,[4] Hamilton,[5] Jacobi[6] and others, will make this principle of least action the most powerful tool to discover the laws of nature.[7,8] It allows, with the same approach, to determine both the equations of particle motion (if we minimize on the trajectories) and the laws of nature (if we minimize on the parameters defining the fields).

However, for the trajectories of particles, this principle is a problem for many scientists as recalled by Henri Poincaré, who was nonetheless one of its major users:[9] "*The very statement of the principle of least action has something shocking to the mind. To go from one point to another one, a material molecule, taken away from the action of any force, but constrained to move on a surface, will follow the geodesic line, i.e. the shortest path. It seems that this molecule knows the point one wants to lead it to, that it anticipates the time needed to reach it along such or such path, and then chooses the most convenient path. In a sense, the statement presents this molecule as a free animated being. It is clear that it would be better to replace it by a less shocking statement where, as philosophers would say, the final causes would not appear to replace the efficient ones.*"

We will see that the difficulties of interpretation of the action come from the existence of two actions corresponding to two different boundary conditions: the classical action (or Euler-Lagrange action) $S_{cl}(\mathbf{x},t;\mathbf{x}_0)$, which links the initial position $\mathbf{x}_0$ and its position $\mathbf{x}$ at time t, and the Hamilton-Jacobi action $S(\mathbf{x},t)$, which links a family of particles of initial action $S_0(\mathbf{x})$ to their various positions $\mathbf{x}$ at time t. In the Euler-Lagrange case, the initial velocity is unknown and in the Hamilton-Jacobi case, the initial position is unknown.

In section 2, we recall the main properties of Euler-Lagrange and Hamilton-Jacobi ac-

tions. In section 3, we see that the Euler-Lagrange action can be considered as the elementary solution of the Hamilton-Jacobi equation in a new branch of nonlinear mathematics, the Minplus analysis. In section 4, we show that the Hamilton-Jacobi action satisfies the principle of least action, contrary to the Euler-Lagrange action, giving a response to the scientists' embarrassment concerning the interpretation of this principle. Finally, in section 5, we use the relation between the Hamilton-Jacobi and Euler-Lagrange actions to study the convergence of quantum mechanics, when the Planck constant tends to 0, for a particular class of quantum systems, the statistical semiclassical case.

II. THE EULER-LAGRANGE AND HAMILTON-JACOBI ACTIONS

Let us consider a system evolving from the position $\mathbf{x}_0$ at initial time to the position $\mathbf{x}$ at time t; let $\mathbf{x}$(s) and $\mathbf{u}$(s) be its position and its velocity at each time $s \in [0, t]$. We have:

$$\frac{d\mathbf{x}\,(s)}{ds} = \mathbf{u}(s) \qquad \text{for } s \in [0, t] \tag{1}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t) = \mathbf{x}. \tag{2}$$

If $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ is the Lagrangian of the system, the Euler-Lagrange action functional is defined by

$$J_{EL}(\mathbf{u}(.)) = \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s)ds \tag{3}$$

where the evolution of $\mathbf{x}(s)$ depends on $\mathbf{u}(s)$ through equations (1) (2).

When the two positions $\mathbf{x}_0$ and $\mathbf{x}$ are given, *the Euler-Lagrange action* $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the function which realizes the minimum (or more generally an extremum[10]) of the Euler-Lagrange action on the velocity field $\mathbf{u}(.)$ for all trajectories from $(\mathbf{x}_0, 0)$ to ($\mathbf{x}$,t):

$$S_{cl}(\mathbf{x}, t; \mathbf{x}_0) = \min_{\mathbf{u}(.)} J_{EL}(\mathbf{u}\,(.)) = \min_{\mathbf{u}(s), 0 \leq s \leq t} \left\{ \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s)ds \right\}, \tag{4}$$

the minimum of (4) is taken on the continuous controls $\mathbf{u}(s)$, $s \in [0, t]$, with the state $\mathbf{x}(s)$ given by the equations (1)(2).

The solution $(\widetilde{\mathbf{u}}(s), \widetilde{\mathbf{x}}(s))$ of (4), if the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ is twice differentiable, satisfies the Euler-Lagrange equations on the interval $[0, t]$:[11]

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) - \frac{\partial L}{\partial \mathbf{x}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) = 0 \qquad (0 \leq s \leq t) \tag{5}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t) = \mathbf{x}. \tag{6}$$

For a nonrelativistic particle in a linear potential field with the Lagrangian $L(\mathbf{x},\dot{\mathbf{x}},t) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \mathbf{K}.\mathbf{x}$, the equation (5) yields $\frac{d}{ds}(m\dot{\mathbf{x}}(s)) - \mathbf{K} = 0$. The trajectory which minimizes the action is $\widetilde{\mathbf{x}}(s) = \mathbf{x}_0 + \frac{s}{t}(\mathbf{x}-\mathbf{x}_0) - \frac{\mathbf{K}}{2m}ts + \frac{\mathbf{K}}{2m}s^2$, and the Euler-Lagrange action is equal to $S_{cl}(\mathbf{x},t;\mathbf{x}_0) = m\frac{(\mathbf{x}-\mathbf{x}_0)^2}{2t} + \frac{K.(\mathbf{x}+\mathbf{x}_0)}{2}t - \frac{K^2}{24m}t^3$.

More generally, in the case of a nonrelativistic particle with the Lagrangian $L(\mathbf{x},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{x},t)$, the Euler-Lagrange action yields the velocities of the two extremities of the trajectory:

$$\mathbf{v}_0 = \dot{\mathbf{x}}(o) = -\frac{1}{m}\frac{\partial S_{cl}}{\partial \mathbf{x}_0}(\mathbf{x},t;\mathbf{x}_0) \qquad \textit{and} \qquad \mathbf{v}(\mathbf{x},t;\mathbf{x}_0) = \dot{\mathbf{x}}(t) = \frac{1}{m}\frac{\partial S_{cl}}{\partial \mathbf{x}}(\mathbf{x},t;\mathbf{x}_0). \tag{7}$$

For the above exemple, we find:

$$\mathbf{v}_0 = \dot{\mathbf{x}}(o) = \frac{\mathbf{x}-\mathbf{x}_0}{t} - \frac{Kt}{2m} \qquad \textit{and} \qquad \mathbf{v}(\mathbf{x},t;\mathbf{x}_0) = \dot{\mathbf{x}}(t) = \frac{\mathbf{x}-\mathbf{x}_0}{t} + \frac{Kt}{2m}.$$

Let us consider now that an initial action $S_0(\mathbf{x})$ is given, then the Hamilton-Jacobi functional from x_0 is defined by:[12]

$$J_{HJ}(\mathbf{u}(.)) = S_0(\mathbf{x}_0) + \int_0^t L(\mathbf{x}(s),\mathbf{u}(s),s)\ ds = S_0(\mathbf{x}_0) + J_{EL}(\mathbf{u}(.)) \tag{8}$$

where the evolution of $\mathbf{x}(s)$ depends on $\mathbf{u}(s)$ through equations (1) (2).

The Hamilton-Jacobi action $S(\mathbf{x},t)$ is the function which realizes the minimum of the Hamilton-Jacobi action functional on the trajectories arriving in $\mathbf{x}$ at time t:

$$S(\mathbf{x},t) = \min_{\mathbf{x}_0;\mathbf{u}(.)} J_{HJ}(\mathbf{u}\,(.)) = \min_{\mathbf{x}_0;\mathbf{u}(s),0\leq s\leq t}\left\{S_0\left(\mathbf{x}_0\right) + \int_0^t L(\mathbf{x}(s),\mathbf{u}(s),s)ds\right\} \tag{9}$$

the minimum of (9) is taken on all initial positions $\mathbf{x}_0$, on the controls $\mathbf{u}(s)$, $s\in[0,t]$, with the state $\mathbf{x}(s)$ given by the equations (1)(2).

Because the term $S_0(\mathbf{x}_0)$ has no effect in (9) for the minimization on $\mathbf{u}(s)$, we deduce the following relation between the Hamilton-Jacobi action and Euler-Lagrange action:

$$S(\mathbf{x},t) = \min_{\mathbf{x}_0}(S_0\left(\mathbf{x}_0\right) + S_{cl}(\mathbf{x},t;\mathbf{x}_0)). \tag{10}$$

For a particle in a linear potential $V(\mathbf{x}) = -\mathbf{K}.\mathbf{x}$ with the initial action $S_0(\mathbf{x}) = m\mathbf{v}_0\cdot\mathbf{x}$, the Hamilton-Jacobi action is equal to $S\left(\mathbf{x},t\right) = m\mathbf{v}_0\cdot\mathbf{x} - \frac{1}{2}m\mathbf{v}_0^2t + \mathbf{K}.\mathbf{x}t - \frac{1}{2}\mathbf{K}.\mathbf{v}_0t^2 - \frac{\mathbf{K}^2t^3}{6m}$.

The Hamilton-Jacobi action $S(\mathbf{x},t)$ defined by (9) can be decomposed into

$$S(\mathbf{x},t) = \min_{\mathbf{x}_0;\mathbf{u}(s),0\leq s\leq t}\left\{S_0\left(\mathbf{x}_0\right) + \int_0^{t-dt} L(\mathbf{x}(s),\mathbf{u}(s),s)ds + \int_{t-dt}^t L(\mathbf{x}(s),\mathbf{u}(s),s)ds\right\}$$

and then satisfies the optimality equation:

$$S(\mathbf{x},t)=\min_{\mathbf{u}(s),t-dt\leq s\leq t}\left\{S(\mathbf{x}-\int_{t-dt}^{t}\mathbf{u}(s)ds,t-dt)+\int_{t-dt}^{t}L(\mathbf{x}(s),\mathbf{u}(s),s)ds\right\}.$$

If we assume S to be differentiable for $\mathbf{x}$ and t, L differentiable for $\mathbf{x}$, $\mathbf{u}$ and t, and $\mathbf{u}$(s) continuous, this equation becomes:

$$0=\min_{\mathbf{u}(t)}\left\{-\frac{\partial S}{\partial\mathbf{x}}\left(\mathbf{x},t\right)\mathbf{u}\left(t\right)dt-\frac{\partial S}{\partial\mathbf{t}}\left(\mathbf{x},t\right)dt+L(\mathbf{x},\mathbf{u}(t),t)dt+\circ\left(dt\right)\right\}$$

and in dividing by dt and letting dt tend towards 0^{+},

$$\frac{\partial S}{\partial\mathbf{t}}\left(\mathbf{x},t\right)=\min_{\mathbf{u}}\left\{L(\mathbf{x},\mathbf{u},t)-\mathbf{u}\cdot\frac{\partial S}{\partial\mathbf{x}}\left(\mathbf{x},t\right)\right\}.$$

We recall that at all convex functions $f(\mathbf{u}):\mathbf{u}\in\mathbb{R}^{n}\rightarrow\mathbb{R}$ we can associate its *Fenchel-Legendre transform* $\widehat{f}(\mathbf{r}):\mathbf{r}\in\mathbb{R}^{n}\rightarrow\mathbb{R}$ defined by $\widehat{f}(\mathbf{r})=\max_{\mathbf{u}\in\mathbb{R}^{n}}(\mathbf{r}\cdot\mathbf{u}-f(\mathbf{u}))$. The Hamiltonian $H(\mathbf{x},\mathbf{p},t)$ is then the Fenchel-Legendre transform of the Lagrangian $L(\mathbf{x},\mathbf{u},t)$ for the variable $\mathbf{u}$.

Then, the Hamilton-Jacobi action satisfies the Hamilton-Jacobi equations:

$$\frac{\partial S}{\partial t}+H(\mathbf{x},\frac{\partial S}{\partial\mathbf{x}},t)=0\tag{11}$$

$$S\left(\mathbf{x},0\right)=S_{0}\left(\mathbf{x}\right).\tag{12}$$

For the Lagrangian $L(\mathbf{x},\dot{\mathbf{x}},t)=\frac{1}{2}m\dot{\mathbf{x}}^{2}-V(\mathbf{x},t)$, $H(\mathbf{x},\nabla S,t)=\max_{\mathbf{v}}(\mathbf{v}.\nabla S-\frac{1}{2}m\mathbf{v}^{2}+V(\mathbf{x},t))$, we have for the optimum $m\mathbf{v}=\nabla S$. We deduce $H(\mathbf{x},\nabla S)=\frac{1}{2m}\left(\nabla S\right)^{2}+V(\mathbf{x},t)$ and the well known result:

The velocity of a nonrelativistic classical particle in a potential field is given for each point $(\mathbf{x},t)$ *by*:

$$\mathbf{v}\left(\mathbf{x},t\right)=\frac{\nabla S\left(\mathbf{x},t\right)}{m}\tag{13}$$

where $S\left(\mathbf{x},t\right)$ *is the Hamilton-Jacobi action, a solution to the Hamilton-Jacobi equations:*

$$\frac{\partial\mathcal{S}}{\partial t}+\frac{1}{2m}(\nabla\mathcal{S})^{2}+V(\mathbf{x},t)=0\tag{14}$$

$$\mathcal{S}(\mathbf{x},0)=\mathcal{S}_{0}(\mathbf{x}).\tag{15}$$

Equation (13) shows that the solution $S\left(\mathbf{x},t\right)$ to the Hamilton-Jacobi equations yields the velocity field for each point $(\mathbf{x},t)$ from the velocity field $\frac{\nabla S_{0}(\mathbf{x})}{m}$ at initial time. In particular, if at initial time, we know the initial position $\mathbf{x}_{init}$ of a particle, its velocity at this time is

equal to $\frac{\nabla S_0(\mathbf{x}_{init})}{m}$. From the solution $S(\mathbf{x},t)$ to the Hamilton-Jacobi equations, we deduce with (13) the trajectories of the particle. The Hamilton-Jacobi action $S(\mathbf{x},t)$ is then a field which "pilots" the particle.

Let us recall how the Hamilton-Jacobi action allows to find Newton's second law of motion.[13] First, we take the gradient of the Hamilton-Jacobi equation (14)$\frac{\partial^2 \mathcal{S}}{\partial t \partial x_i} + \frac{1}{m}\sum_j \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j}\frac{\partial \mathcal{S}}{\partial x_j} + \frac{\partial V}{\partial x_i} = 0$. Second, we remark that $\frac{d}{dt}\left(\frac{\partial \mathcal{S}}{\partial x_i}\right) = \frac{\partial^2 \mathcal{S}}{\partial t \partial x_i} + \sum_j \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} v_j$ and with the equation (13) where $m\mathbf{v} = \nabla S$, we conclude $\frac{d}{dt}(m\mathbf{v}) = -\nabla V$.

III. THE EULER-LAGRANGE AND HAMILTON-JACOBI ACTIONS IN MINPLUS ANALYSIS

There is a new branch of mathematics, the Minplus analysis, which studies nonlinear problems through a linear approach, cf. Maslov[14,15] and Gondran[16,17]. The idea is to substitute the usual scalar product $\int_X f(x)g(x)dx$ by the Minplus scalar product:

$$(f,g) = \inf_{x \in X}\{f(x) + g(x)\} \tag{16}$$

In the scalar product we replace the field of the real number $(\mathbb{R}, +, \times)$ with the algebraic structure *Minplus* $(\mathbb{R} \cup \{+\infty\}, \min, +)$, i.e. the set of real numbers (with the element infinity $\{+\infty\}$) endowed with the operation Min (minimum of two reals), which remplaces the usual addition, and with the operation $+$ (sum of two reals), which remplaces the usual multiplication. The element $\{+\infty\}$ corresponds to the neutral element for the operation Min, $\mathrm{Min}(\{+\infty\}, a) = a \ \forall a \in \mathbb{R}$.

This approach bears a close similarity to *the theory of distributions for the nonlinear case*; here, the operator is "linear" and continuous with respect to the Minplus structure, though *nonlinear* with respect to the classical structure $(\mathbb{R}, +, \times)$. In this Minplus structure, the Hamilton-Jacobi equation is linear, because if $S_1(\mathbf{x}, t)$ and $S_2(\mathbf{x}, t)$ are solutions to (14), then $\min\{\lambda + S_1(\mathbf{x}, t), \mu + S_2(\mathbf{x}, t)\}$ is also solution to the Hamilton-Jacobi equation (14).

The analogue to the Dirac distribution $\delta(\mathbf{x})$ in Minplus analysis is the nonlinear distribution $\delta_{\min}(\mathbf{x}) = \{0 \ if \ \mathbf{x} = \mathbf{0}, +\infty \ if \ not\}$. With this nonlinear Dirac distribution, we can define elementary solutions as in classical distribution theory. In particular, we have:

The classical Euler-Lagrange action $S_{cl}(\boldsymbol{x}, t; \boldsymbol{x}_0)$ *is the elementary solution to the*

Hamilton-Jacobi equations (11)(12) in the Minplus analysis with the initial condition

$$S(\mathbf{x},0)=\delta_{\min}(\mathbf{x}-\mathbf{x}_0)=\{0 \;\; if \;\; \mathbf{x}=\mathbf{x}_0, \;\; +\infty \;\; if \;\; not\}.$$

The Hamilton-Jacobi action $S(\mathbf{x},t)$ is then given by the Minplus integral:

$$S(\mathbf{x},t)=\inf_{\mathbf{x}_0}\{S_0(\mathbf{x}_0)+S_{cl}(\mathbf{x},t;\mathbf{x}_0)\}$$

in analogy with the solution of the heat transfer equation given by the classical integral:

$$S(x,t)\;=\;\int S_0\left(x_0\right)\frac{1}{2\sqrt{\pi t}}\;e^{-\dfrac{\left(x-x_0\right)^2}{4t}}\,dx_0.$$

In this Minplus analysis, the Legendre-Fenchel transform is the analogue to the Fourier transform. This transform is known to have many applications in Physics: this is the one which sets the correspondence between the Lagrangian and the Hamiltonian of a physical system; which sets the correspondence between microscopic and macroscopic models; which is also at the basis of multifractal analysis relevant to modeling turbulence in fluid mechanics.[17]

IV. THE PRINCIPLE OF LEAST ACTION

Equation (9) shows that, among the trajectories which can reach $(\mathbf{x},t)$ from a position at the initial time whose initial velocity field is known, Nature chooses the velocity which minimizes (or realizes the extremum) of the Hamilton-Jacobi functional. Then, the principle of least action defines the velocity field at time t: ($\mathbf{v}(\mathbf{x},t)=\frac{\nabla S(\mathbf{x},t)}{m}$). For the Lagrangian $L(\mathbf{x},\dot{\mathbf{x}},t)=\frac{1}{2}m\dot{\mathbf{x}}^2-\mathbf{K}.\mathbf{x}$, we find $\mathbf{v}(\mathbf{x},t)=\mathbf{v}_0+\mathbf{K}t/m$. The Hamilton-Jacobi action $S(\mathbf{x},t)$ does not solve only a given problem with a single initial condition $\left(\mathbf{x}_0,\frac{\nabla S_0(\mathbf{x}_0)}{m}\right)$, but a set of problems with an infinity of initial conditions, all the couples $\left(\mathbf{y},\frac{\nabla S_0(\mathbf{y})}{m}\right)$.

Does the Euler-Lagrange action correspond to the principle of least action? The answer seems positive from equation (4). But in fact, in the absence of an initial velocity field as in the Hamilton-Jacobi action, the Euler-Lagrange action answers a problem posed by the observer, and not by Nature: "If we see that a particle in $\mathbf{x}_0$ at the initial time arrives in $\mathbf{x}$ at time t, what was its initial velocity $\mathbf{v}_0$?" To solve this problem, the observer must solve the Euler-Lagrange equations (5,6) which are difficult because they concern the entire

trajectory. We have seen that this velocity is given by equation (7): $\mathbf{v}_0 = -\frac{1}{m}\frac{\partial S_{cl}}{\partial \mathbf{x}_0}(\mathbf{x},t;\mathbf{x}_0)$. This is an *a posteriori* point of view.

Contrary to the Euler-Lagrange action, the Hamilton-Jacobi action answers the following problem:"If we know the action (or the velocity field) at the initial time, can we determine the action (or the velocity field) at each later time?" This problem is solved from time to time by the evolution equation (11) or (14) which is local. This is an *a priori* point of view. It is the problem solved by Nature with the principle of least action. The Euler-Lagrange action, which is an elementary solution to the Hamilton-Jacobi equation, seems to satisfy this principle. But, only the Hamilton-Jacobi action satisfies the principle of least action. It is a clear answer to the Poincaré question on the interpretation of this principle.

V. LIMIT OF THE SCHRÖDINGER EQUATION IN THE STATISTICAL SEMI-CLASSICAL CASE

Equation (10) between the Hamilton-Jacobi and Euler-Lagrange actions will allow us to study the convergence of quantum mechanics, when the Planck constant tends to 0, for a particular class of quantum systems.

Let us consider the wave function solution to the Schrödinger equation $\Psi(\mathbf{x},t)$:

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\Psi + V(\mathbf{x},t)\Psi \tag{17}$$

$$\Psi(\mathbf{x},0) = \Psi_0(\mathbf{x}). \tag{18}$$

With the variable change $\Psi(\mathbf{x},t) = \sqrt{\rho^\hbar(\mathbf{x},t)}\exp(i\frac{S^\hbar(\mathbf{x},t)}{\hbar})$, the quantum density $\rho^\hbar(\mathbf{x},t)$ and the quantum action $S^\hbar(\mathbf{x},t)$ depend on the parameter $\hbar$. The Schrödinger equation can be decomposed into Madelung equations[18] (1926):

$$\frac{\partial S^\hbar(\mathbf{x},t)}{\partial t} + \frac{1}{2m}(\nabla S^\hbar(\mathbf{x},t))^2 + V(\mathbf{x},t) - \frac{\hbar^2}{2m}\frac{\triangle\sqrt{\rho^\hbar(\mathbf{x},t)}}{\sqrt{\rho^\hbar(\mathbf{x},t)}} = 0 \tag{19}$$

$$\frac{\partial \rho^\hbar(\mathbf{x},t)}{\partial t} + div(\rho^\hbar(\mathbf{x},t)\frac{\nabla S^\hbar(\mathbf{x},t)}{m}) = 0 \qquad \forall(\mathbf{x},t) \tag{20}$$

with initial conditions

$$\rho^\hbar(\mathbf{x},0) = \rho_0^\hbar(\mathbf{x}) \qquad and \qquad S^\hbar(\mathbf{x},0) = S_0^\hbar(\mathbf{x}). \tag{21}$$

Here, we study the convergence of the density $\rho^\hbar(\mathbf{x},t)$ and the action $S^\hbar(\mathbf{x},t)$, when $\hbar$ tends to 0, for a particular preparation of the particles. A quantum system is prepared in *the*

statistical semi-classical case[19,20] if its initial probability density $\rho_0^\hbar(\mathbf{x})$ and its initial action $S_0^\hbar(\mathbf{x})$ are regular functions $\rho_0(\mathbf{x})$ and $S_0(\mathbf{x})$ not depending on $\hbar$, and its interaction with the potential field $V(\mathbf{x},t)$ can be described classically. It is the case of a set of particles that are non-interacting and prepared in the same way: a free particles beam in a linear potential, an electronic or C_{60} beam in the Young's slits diffraction, an atomic beam in the Stern and Gerlach experiment. Then, we have the following result:[19,20]

For particles in the statistical semi-classical case, the probability density $\rho^\hbar(\boldsymbol{x},t)$ and the action $S^\hbar(\boldsymbol{x},t)$, solutions to the Madelung equations (19)(20)(21), converge, when $\hbar \to 0$, to the classical density $\rho(\boldsymbol{x},t)$ and the classical action $S(\boldsymbol{x},t)$, solutions to the statistical Hamilton-Jacobi equations:

$$\frac{\partial S\left(\mathbf{x},t\right)}{\partial t}+\frac{1}{2m}(\nabla S(\mathbf{x},t))^2+V(\mathbf{x},t)=0 \tag{22}$$

$$\frac{\partial \rho\left(\mathbf{x},t\right)}{\partial t}+div\left(\rho\left(\mathbf{x},t\right)\frac{\nabla S\left(\mathbf{x},t\right)}{m}\right)=0 \qquad \forall\left(\mathbf{x},t\right) \tag{23}$$

$$\rho(\mathbf{x},0)=\rho_0(\mathbf{x}) \qquad and \qquad S(\mathbf{x},0)=S_0(\mathbf{x}). \tag{24}$$

We will demonstrate in the case where the wave function $\Psi(\mathbf{x},t)$ at time t is written as a function of the initial wave function $\Psi_0(\mathbf{x})$ by the Feynman paths integral formula[22] (p. 58):

$$\Psi(\mathbf{x},t)=\int F(t,\hbar)\exp(\frac{i}{\hbar}S_{cl}(\mathbf{x},t;\mathbf{x}_0)\Psi_0(\mathbf{x}_0)d\mathbf{x}_0$$

where $F(t,\hbar)$ is an independent function of $\mathbf{x}$ and of $\mathbf{x}_0$ and where $S_{cl}(\mathbf{x},t;\mathbf{x}_0)$ is the classical action. In the statistical semi-classical case, the wave function is written $\Psi(\mathbf{x},t) = F(t,\hbar)\int\sqrt{\rho_0(\mathbf{x}_0)}\exp(\frac{i}{\hbar}(S_0(\mathbf{x}_0)+S_{cl}(\mathbf{x},t;\mathbf{x}_0))d\mathbf{x}_0$. The theorem of the stationary phase[22] shows that, if $\hbar$ tends towards 0, we have $\Psi(\mathbf{x},t)\sim\exp(\frac{i}{\hbar}min_{\mathbf{x}_0}(S_0(\mathbf{x}_0)+S_{cl}(\mathbf{x},t;\mathbf{x}_0))$, that is to say that the quantum action $S^h(\mathbf{x},t)$ converges to the function

$$S(\mathbf{x},t)=min_{\mathbf{x}_0}(S_0(\mathbf{x}_0)+S_{cl}(\mathbf{x},t;\mathbf{x}_0))$$

which is the solution to the Hamilton-Jacobi equation (22) with the initial condition (24). Moreover, as the quantum density $\rho^h(\mathbf{x},t)$ verifies the continuity equation (20), we deduce, since $S^h(\mathbf{x},t)$ tends towards $S(\mathbf{x},t)$, that $\rho^h(x,t)$ converges to the classical density $\rho(\mathbf{x},t)$, which satisfies the continuity equation (23). We obtain both announced convergences.

If we consider the system with initial conditions $\rho_0^\hbar(\mathbf{x})=\rho_0(\mathbf{x})=(2\pi\sigma_0^2)^{-\frac{3}{2}}e^{-\frac{(\mathbf{x}-\zeta_0)^2}{2\sigma_0^2}}$ and $S_0^\hbar(\mathbf{x})=S_0(\mathbf{x})=m\mathbf{v}_0\cdot\mathbf{x}$ in a linear potential field $V(\mathbf{x})=-\mathbf{K}\cdot\mathbf{x}$, where σ_0, $\mathbf{v}_0$, ζ_0 and

$\mathbf{K}$ are constants independent of $\hbar$, the density $\rho^{\hbar}(\mathbf{x},t)$ and the action $S^{\hbar}(\mathbf{x},t)$ are equal to[21]
$\rho^{\hbar}(\mathbf{x},t) = (2\pi\sigma_{\hbar}^2(t))^{-\frac{3}{2}} exp - \frac{\left(\mathbf{x}-\zeta_0-\mathbf{v}_0 t-\mathbf{K}\frac{t^2}{2m}\right)^2}{2\sigma_{\hbar}^2(t)}$ and

$$S^{\hbar}(\mathbf{x},t) = -\frac{3\hbar}{2}tg^{-1}(\hbar t/2m\sigma_0^2) - \frac{1}{2}m\mathbf{v}_0^2 t + m\mathbf{v}_0\cdot\mathbf{x} + \mathbf{K}\cdot\mathbf{x}t - \frac{1}{2}\mathbf{K}\cdot\mathbf{v}_0 t^2 - \frac{\mathbf{K}^2 t^3}{6m} + \frac{\left(\mathbf{x}-\zeta_0-\mathbf{v}_0 t-\mathbf{K}\frac{t^2}{2m}\right)^2 \hbar^2 t}{8m\sigma_0^2\sigma_{\hbar}^2(t)}$$

with $\sigma_{\hbar}(t) = \sigma_0\left(1+(\hbar t/2m\sigma_0^2)^2\right)^{\frac{1}{2}}$. When $\hbar \to 0$, $\sigma_{\hbar}(t)$ converges to σ_0 and the density $\rho^{\hbar}(\mathbf{x},t)$ and the action $S^{\hbar}(\mathbf{x},t)$ converge to $\rho(\mathbf{x},t) = (2\pi\sigma_0^2)^{-\frac{3}{2}} e^{-\frac{\left(\mathbf{x}-\zeta_0-\mathbf{v}_0 t-\mathbf{K}\frac{t^2}{2m}\right)^2}{2\sigma_0^2}}$ and $S(\mathbf{x},t) = -\frac{1}{2}m\mathbf{v}_0^2 t + m\mathbf{v}_0\cdot\mathbf{x} + \mathbf{K}\cdot\mathbf{x}t - \frac{1}{2}\mathbf{K}\cdot\mathbf{v}_0 t^2 - \frac{\mathbf{K}^2 t^3}{6m}$ which are solutions to statistical Hamilton-Jacobi equations (22)(23)(24).

The statistical Hamilton-Jacobi equations correspond to a set of independent classical particles, in a potential field $V(\mathbf{x},t)$, and for which we only know at the initial time the probability density $\rho_0(\mathbf{x})$ and the velocity $\mathbf{v}(\mathbf{x}) = \frac{\nabla S_0(\mathbf{x},t)}{m}$. These particles are not indistinguishable because, if their initial positions are known, their trajectories will also be known. Nevertheless, these particles will have the same properties as the indistinguishable ones. Thus, if the initial density $\rho_0(\mathbf{x})$ is given, and one randomly chooses N particles, the N! permutations are strictly equivalent and do not correspond to the same configuration as for indistinguishable particles.

For particles prepared in the statistical semi-classical case, the uncertainity about the position of a quantum particle corresponds to an uncertainity about the position of a classical particle, whose initial density alone has been defined. *In classical mechanics, this uncertainity is removed by giving the initial position of the particle. It would not be logical not to do the same in quantum mechanics.* It is then possible to assume that for *the statistical semi-classical case*, a quantum particle is not well described by its wave function. One needs therefore to add its initial position and it becomes natural to introduce the so-called de Broglie-Bohm trajectories[23,24] with the velocity $\mathbf{v}^{\hbar}(\mathbf{x},t) = \frac{1}{m}\nabla S^{\hbar}(\mathbf{x},t)$.

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